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Abstract

This paper presents an approach to improve Prony's method of identifying a linear time-invariant system. The method is based on a nonlinear transformation of parameters, which leads to data averaging. The method yields better results than a direct application of the least squares approach to Prony's method. A numerical example is given to demonstrate the improvement attained by the new algorithm. Signals are assumed to be contaminated by zero-mean Gaussian white noise.

Introduction

In the past few decades, numerous research activities have been devoted to the identification of linear time-invariant systems (Åström et al. 1971) Among some well-known approaches are Prony-based methods (Lacroix 1973, Khatwani et al. 1975, Kumaresan 1990, Pierre et al. 1992, Hietpas 1994, Pierre et al. 1995), Mellin deconvolution (Prost et al. 1976) numerical computation of Laplace transform (Unnikrishnan 1980) and genetic algorithm (Kristinsson et al. 1992). For control applications, it is often required to determine a system transfer function with certain accuracy from a limited amount of sampled data (Kumaresan 1990 and Pierre 1995).

This paper presents an improvement of Prony's method of transfer function identification. An algorithm is developed based on the introduction of finite differences. The result is a nonlinear transformation of the parameters to be identified. A numerical study of an example extracted from the literature (Lacroix 1973) is presented. The different cases of varying sampling intervals are considered. Results with recursive implementation are compared with those of the conventional Prony method.

Materials and Methods

Prony's Method

Consider a linear time-invariant system in Figure 1

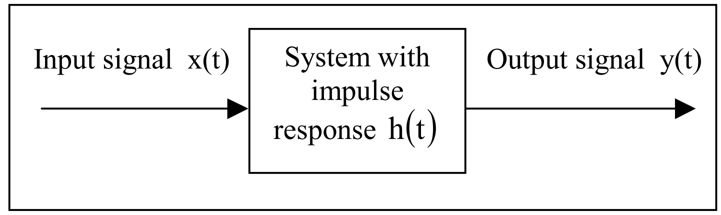

Figure 1. Linear time-invariant system with impulse response h(t).

with the following rational transfer function:

$$H(s) = k\frac{\prod_{j=1}^{m}(s - z_j)}{\prod_{i=1}^{n}(s - p_i)} \tag{1}$$

where m < n, and the poles p_i are assumed to be distinct. On taking partial fraction expansion, H(s) in Eq. (1) becomes

$$H(s) = \sum_{i=1}^{n}\frac{c_i}{s - p_i} \tag{2}$$

Inverse Laplace transform of Eq. (2) gives the impulse response of the system as a sum of exponential terms:

$$h(t) = \sum_{i=1}^{n} c_i e^{p_i t} \tag{3}$$

The objective of the following is to identify the unknown exponents p_i and the unknown weights c_i. Let the system impulse response h(t) be sampled with a sampling interval of Δt to yield N data points. Then, with $\phi_i \triangleq e^{p_i \Delta t}$, Eq. (3) gives

$$h(k\Delta t) \triangleq y_k = \sum_{i=1}^{n} c_i \phi_i^k, \quad k = 0, 1, \cdots, N-1 \tag{4}$$

which, written in full, yields the following system of N equations in 2n unknowns:

$$\begin{array}{lllll} c_1 & +c_2 & +\cdots & +c_n & =y_0 \\ c_1\phi_1 & +c_2\phi_2 & +\cdots & +c_n\phi_n & =y_1 \\ c_1\phi_1^2 & +c_2\phi_2^2 & +\cdots & +c_n\phi_n^2 & =y_2 \\ & \vdots & & & \\ c_1\phi_1^{N-1} & +c_2\phi_2^{N-1} & +\cdots & +c_n\phi_n^{N-1} & =y_{N-1} \end{array} \tag{5}$$

If more samples than necessary are taken to reduce the effect of noise, i.e. N > 2n, the unknown system can be identified by solving the above set of overdetermined nonlinear algebraic equations. From difference equation theory, y_k in Eq. (4) is the general solution of the nth order difference equation:

$$\begin{gathered} y_{k+n}+\tilde{c}_{n-1}y_{k+n-1}+\cdots+\tilde{c}_1 y_{k+1}+\tilde{c}_0 y_k=0 \\ k=0,1,\cdots,N-n-1 \end{gathered} \tag{6}$$

Thus, ϕ_i in Eq. (4) must satisfy the characteristic equation of Eq. (6):

$$\phi^n+\tilde{c}_{n-1}\phi^{n-1}+\cdots+\tilde{c}_1\phi+\tilde{c}_0=0 \tag{7}$$

Equation (6) can be expressed in matrix notation as

$$\mathbf{A}\tilde{\mathbf{c}}=-\mathbf{y_1} \tag{8}$$

where

$$\mathbf{A}=\begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & & \vdots \\ y_{N-n-1} & y_{N-n} & \cdots & y_{N-2} \end{bmatrix} \tag{9}$$

$$\tilde{\mathbf{c}}=\begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{n-1} \end{bmatrix} \tag{10}$$

$$\mathbf{y_1}=\begin{bmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{N-1} \end{bmatrix} \tag{11}$$

The coefficient vector $\tilde{\mathbf{c}}$ can be obtained in the least-squares sense by

$$\tilde{\mathbf{c}}=-(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y_1} \tag{12}$$

with the assumption that **A** is of full rank.

After $\tilde{\mathbf{c}}$ have been determined, the ϕ_i in Eq. (5) are then calculated as the n roots of Eq. (7). They are either real or appear in complex conjugate pairs. With known ϕ_i, Eq. (5) can be written in the following form

$$\mathbf{\Phi c}=\mathbf{y} \tag{13}$$

where

$$\mathbf{\Phi}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & \vdots & & \vdots \\ \phi_1^{N-1} & \phi_2^{N-1} & \cdots & \phi_n^{N-1} \end{bmatrix} \tag{14}$$

$$\mathbf{c}=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{15}$$

$$\mathbf{y}=\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} \tag{16}$$

Once again, using the least-squares approach, **c** in Eq. (13) can be evaluated. By taking the natural logarithm of $\phi_i=e^{p_i\Delta t}$(1), the system poles p_i can be determined. Therefore, all the n unknown weights and exponents in Equation (3) are completely obtained. The unknown system H(s) in Eq. (2) is hence identified.

Improvement of Prony's Method using Finite Differences

The following is a modification of Prony's algorithm based on the idea that averaging multiple captures of a signal will reduce the effect of noise in the final result. In Eq. (6), y_k are known data values, and the unknown parameters to be estimated are $\tilde{c}_i$. The basic idea of the proposed method is to re-write Eq. (6) into a different form by introducing finite differences as follows. Consider the sequence y_k, $k = 0, 1,\cdots, N-1$. Define the finite difference operator Δ as

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(1) Here, it is assumed that the sampling interval Δt does not exceed an upper limit of $\Delta t \le \dfrac{\pi}{\max|\mathrm{Im}\, p_i|}$ so that $\ln\phi_i$ will be a single-valued function (see Lacroix 1973).

$$\begin{aligned} \Delta^0 y_k &\triangleq y_k \triangleq y_k^{(0)} \\ \Delta^1 y_k &\triangleq y_k - y_{k-1} \triangleq y_k^{(1)} \\ \Delta^2 y_k = \Delta(\Delta y_k) &= \Delta y_k^{(1)} \triangleq y_k^{(1)} - y_{k-1}^{(1)} \triangleq y_k^{(2)} \\ &\vdots \\ \Delta^n y_k &\triangleq \Delta(\Delta^{n-1} y_k) \triangleq y_k^{(n)} \end{aligned} \tag{17}$$

It can be shown (see Appendix) that Eq. (6) can be rewritten in terms of the finite differences defined above as

$$\begin{aligned} b_n y_k^{(n)} + b_{n-1} y_k^{(n-1)} + \cdots + b_1 y_k^{(1)} = -y_k, \\ k = n, n+1, \cdots, N-1 \end{aligned} \tag{18}$$

Here, the finite differences $y_k^{(1)}, \cdots, y_k^{(n)}$ can be calculated from the data values y_k, and the parameters $b_1, b_2, \cdots, b_n$ are to be estimated. The former parameters $\tilde{c}_0, \tilde{c}_1, \cdots, \tilde{c}_{n-1}$ and the new parameters $b_1, b_2, \cdots, b_n$ are related by

$$\tilde{c}_i = \frac{(-1)^{n-i} \sum_{\nu=n-i}^{n} \binom{\nu}{n-i} b_\nu}{1 + \sum_{\nu=1}^{n} b_\nu}, \quad i = 0, 1, \cdots, n-1 \tag{19}$$

Eq. (19) represents a nonlinear parameter transformation between b_i and $\tilde{c}_i$.

Next, we sum up both sides of Eq. (18) from k = n to an arbitrary value of k to yield

$$\begin{aligned} b_n y_k^{(n-1)} + b_{n-1} y_k^{(n-2)} + \cdots b_1 y_k = -\sum_{k1=n}^{k} y_{k1} \\ k = n, n+1, \cdots, N-1 \end{aligned} \tag{20}$$

In obtaining Eq. (20), use has been made of the relation

$$\sum_{k_1=n}^{k} y_{k_1}^{(i)} = y_k^{(i-1)}, \quad i = 1, 2, \cdots, n \tag{21}$$

which is valid provided that (straight-forward to verify)

$$y_{n-1}^{(n-1)} = y_{n-1}^{(n-2)} = \cdots = y_{n-1}^{(0)} = 0 \tag{22}$$

Note that assumption (22) is equivalent to "zero initial-conditions", i.e.

$$y_0 = y_1 = y_2 = \cdots = y_{n-1} = 0 \tag{23}$$

Summing up Eq. (20) from k=n to any k yields

$$\begin{aligned} & b_n y_k^{(n-2)} + b_{n-1} y_k^{(n-3)} + \cdots + b_2 y_k + b_1 \sum_{k_2=n}^{k} y_{k_2} \\ &= -\sum_{k_2=n}^{k} \sum_{k_1=n}^{k_2} y_{k_1} \qquad k = n, n+1, \cdots, N-1 \end{aligned} \tag{24}$$

A similar summing process is applied to over Eq. (24) to yield the next equation, and this summing process is repeated successively up to a total of n times to give

$$\begin{aligned} & b_n y_k + b_{n-1} \sum_{k_n=n}^{k} y_{k_n} + b_{n-2} \sum_{k_n=n}^{k} \sum_{k_{n-1}=n}^{k_n} y_{k_{n-1}} + \cdots \\ & + b_1 \sum_{k_n=n}^{k} \sum_{k_{n-1}=n}^{k_n} \cdots \sum_{k_2=n}^{k_3} y_{k_2} = -\sum_{k_n=n}^{k} \sum_{k_{n-1}=n}^{k_n} \cdots \sum_{k_1=n}^{k_2} y_{k_1} \\ & k = n, n+1, \cdots, N-1 \end{aligned} \tag{25}$$

Putting Eqs. (20), (24) on up to (25) in reserve order in matrix form, we have

$$\begin{bmatrix} y_k^{(-n+1)} & y_k^{(-n+2)} & \cdots & y_k^{(-1)} & y_k \\ y_k^{(-n+2)} & y_k^{(-n+3)} & \cdots & y_k & y_k^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_k^{(-1)} & y_k & \cdots & y_k^{(n-3)} & y_k^{(n-2)} \\ y_k & y_k^{(1)} & \cdots & y_k^{(n-2)} & y_k^{(n-1)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = -\begin{bmatrix} y_k^{(-n)} \\ y_k^{(-n+1)} \\ \vdots \\ y_k^{(-2)} \\ y_k^{(-1)} \end{bmatrix}, \qquad k = n, n+1, \cdots, N-1 \tag{26}$$

where

$$y_k^{(-i)} \triangleq \sum_{k_i=n}^{k} \sum_{k_{i-1}=n}^{k_i} \cdots \sum_{k_1=n}^{k_2} y_{k_1}, \quad i = 1, 2, \cdots, n \tag{27}$$

The unknown parameters b_i can be solved from the symmetric system (26) for any given time $k = n, n+1, \cdots, N-1$ provided that the square matrix is invertible. Using Eq. (19), the original parameters $\tilde{c}_i$ can be calculated. With known $\tilde{c}_i$, the rest of the procedure follows what is described in Section II.

Results

The two methods described in Sections I and II were applied, respectively, to identify a system with the help of MATHCAD software. The system to be identified is a fifth-order system from (Lacroix A. 1973) with the poles: $p_1 = -1$, $p_{2,3} = -1 \pm j1$, $p_{4,5} = -1 \pm j2$. The unit impulse response for this system is given by

$$h(t) = \frac{5}{2}e^{-t} - \frac{5}{3}e^{(-1+j1)t} - \frac{5}{3}e^{(-1-j1)t} + \frac{5}{12}e^{(-1+j2)t} + \frac{5}{12}e^{(-1-j2)t} \tag{28}$$

The impulse response sampled with 100 points over a period of 6 seconds is shown in Figure 2.

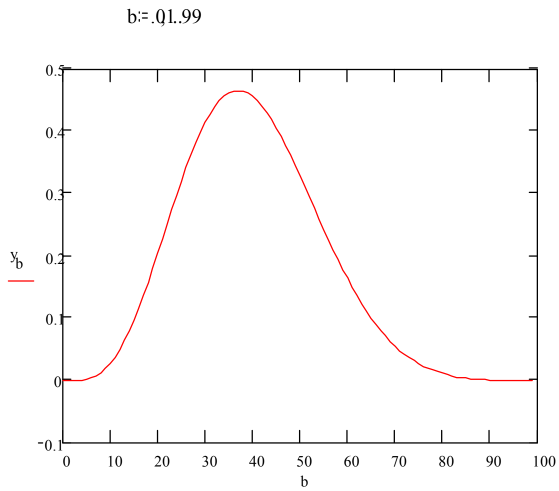

Figure 2. Impulse Response of 5-pole System for 100 Sample Points.

To evaluate the two algorithms, the impulse response is first sampled, and then a Gaussian white noise of zero mean is added onto the sampled data. The standard deviation of the noise added is 3% of the maximum value of the impulse response. Figure 3 shows a typical noise-contaminated impulse response for 100 sample points.

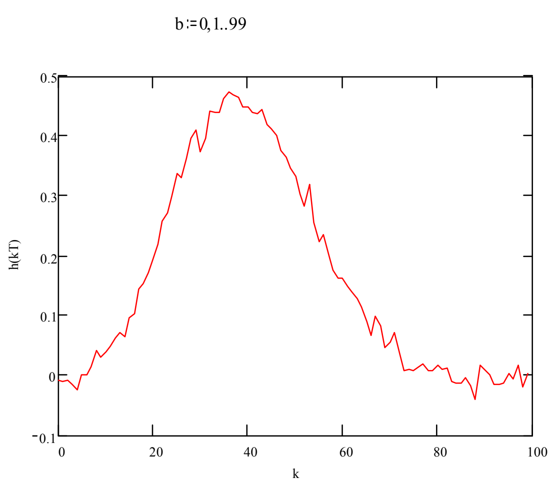

Figure 3. Impulse response of system (28) contaminated with a zero-mean Gaussian white noise 100 sample points.

Table 1 summarizes the identification results of the conventional Prony's method for 40, 70 and 100 sample points, all taken over a period of 6 seconds. Ten randomly generated noise sequences as described above are superimposed in the impulse response h(kΔt). The corresponding sampling intervals are 0.15, 0.0857142 and 0.06 sec, respectively. All the sampling intervals used above comply with the criterion of Footnote 1, because:

$$\frac{\pi}{\max\left|\mathrm{Im}\{p_i\}\right|} = \frac{\pi}{2} = 1.571$$

Table 1. The root-mean-square identification errors for the conventional Prony method when data are contaminated by 10 randomly generated noise sequences

Number of Trials	40 Points	70 Points	100 Points
1st Trial	0.08491	0.08091	0.10630
2nd Trial	0.08145	0.08217	0.11630
3rd Trial	0.08020	0.07869	0.12290
4th Trial	0.06722	0.08521	0.10590
5th Trial	0.06894	0.08244	0.11560
6th Trial	0.07818	0.08553	0.10360
7th Trial	0.07651	0.08070	0.10960
8th Trial	0.07266	0.08194	0.11300
9th Trial	0.07897	0.09079	0.10580
10th Trial	0.07838	0.08047	0.09243
Average	0.07674	0.08289	0.10914

The root-mean-square identification error in Table 1 is defined as

$$e_{rms} \triangleq \sqrt{\frac{\sum_{k=0}^{N-1}\left|\hat{y}_k - y_k\right|^2}{N}} \tag{29}$$

where N is the total number of sample points, and $\hat{y}_k$ is the output of the identified model i.e. the value of y_k as calculated according to Eq. (4) when estimates for c_i and ϕ_i are used.

Table 2 shows the comparison results of the finite-difference method with 40, 70 and 100 points. The same 10 sets of data were used as those in Table 1.

Table 2. The root-mean-square identification errors for the proposed finite-difference method with the same 10 contaminated data sequences as in Table 1.

Number of Trials	40 Points	70 Points	100 Points
1st Trial	0.05004	0.08147	0.07520
2nd Trial	0.05330	0.07437	0.08194
3rd Trial	0.06184	0.06665	0.07280
4th Trial	0.05652	0.07423	0.08463
5th Trial	0.05831	0.07880	0.07205
6th Trial	0.05121	0.05213	0.08637
7th Trial	0.05445	0.08075	0.08533
8th Trial	0.05599	0.07624	0.07260
9th Trial	0.06825	0.06118	0.08131
10th Trial	0.06184	0.08299	0.07310
Average	0.05718	0.07288	0.07853

As can be seen from the comparing of Tables 1 and 2, the proposed finite-difference method outperforms the conventional Prony's method in terms of accuracy. The reductions in the root-mean-square error for the 40-point, 70-point and 100-point cases are, on the average, 25%, 12% and 31%, respectively.

Table 3. The demonstration of a "two-tailed, paired t-test for the establishment of statistically significant data.

		t	df	Sig. (2-tailed)
Pair 1	T1.40 - T2.40	7.427	9	.000
Pair 2	T1.70 - T2.70	2.564	9	.030
Pair 3	T1.100 - T2.100	8.644	9	.000

Table 3 was simulated using SPSS statistical software to compare the before added noise data and after added noise data. Pair 1 of Row 2 in Table 3 was compared before the added noise and after the added noise for 40 points. Pair 2 of Row 3 in Table 3 was compared before the added noise and after added noise for 70 points. Pair 3 of Row 4 in Table 3 was compared before the added noise and after added noise for 100 points. As can be seen from Column 4 of Table 3, to establish the improvement of statistically significant, the compared data have shown that ($p < 0.05$). In specific, for comparison of p value for 40- point and 100 -point are zero. The comparisons of p value for 70-points is only 0.03. Therefore, all the compared data are statistically significant.

Conclusions

A method is presented to improve the performance of the conventional Prony's approach of transfer function identification. The method is based on the introduction of finite differences, resulting in a nonlinear transformation of the system parameters to be identified. A numerical study was conducted on a five-pole system extracted from the literature. The proposed finite-difference method was found to yield consistently more accurate results than the conventional Prony's approach. The motivation of this research is to impact the controllability of linear time-invariant system.

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